EXERCISES

1. Expand $(1 - z)^{-m}$, m a positive integer, in powers of z.

2. Expand $\frac{2z + 3}{z + 1}$ in powers of $z - 1$. What is the radius of convergence?

3. Find the radius of convergence of the following power series:

$$\sum n^p z^n, \sum \frac{z^n}{n!}, \sum n! z^n, \sum q^{n^2} z^n (|q| < 1), \sum z^{n!}$$

4. If $\Sigma a_n z^n$ has radius of convergence R, what is the radius of convergence of $\Sigma a_n z^{2n}$? of $\Sigma a_n^2 z^n$?

5. If $f(z) = \Sigma a_n z^n$, what is $\Sigma n^3 a_n z^n$?

6. If $\Sigma a_n z^n$ and $\Sigma b_n z^n$ have radii of convergence R_1 and R_2, show that the radius of convergence of $\Sigma a_n b_n z^n$ is at least $R_1 R_2$.

7. If $\lim_{n\to\infty} |a_n|/|a_{n+1}| = R$, prove that $\Sigma a_n z^n$ has radius of convergence R.

8. For what values of z is

$$\sum_0^\infty \left(\frac{z}{1 + z}\right)^n$$

convergent?

9. Same question for

$$\sum_0^\infty \frac{z^n}{1 + z^{2n}}.$$

2.5. Abel's Limit Theorem. There is a second theorem of Abel's which refers to the case where a power series converges at a point of the circle of convergence. We lose no generality by assuming that $R = 1$ and that the convergence takes place at $z = 1$.

Theorem 3. *If $\sum_0^\infty a_n$ converges, then $f(z) = \sum_0^\infty a_n z^n$ tends to $f(1)$ as z approaches 1 in such a way that $|1 - z|/(1 - |z|)$ remains bounded.*

Remark. Geometrically, the condition means that z stays in an angle $<180°$ with vertex 1, symmetrically to the part $(-\infty, 1)$ of the real axis. It is customary to say that the approach takes place in a *Stolz angle.*

Proof. We may assume that $\sum_0^\infty a_n = 0$, for this can be attained by adding